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Scaling behaviour of cluster hulls in spiral site percolation

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Abstract. The scaling behaviour of the percolation hull of large clusters in the case of rotationally constrained site percolation on the square and triangular lattices is studied. The single cluster growth method has been used to determine the values of some critical exponents. Evidence is obtained for a scaling form of the hull size distribution function.

1. Introduction

Percolation under rotational constraint, referred to as 'spiral percolation', has recently been studied (Santra and Bose 1981, 1992) on the square and triangular lattices. In spiral percolation, each path proceeds either straight or in a specific rotational direction, say clockwise. Percolation occurs if a cluster obeying the rotational constraint spans the underlying lattice. The critical percolation probability p_c has been determined using the binary search method. Different techniques like finite-size scaling, Monte Carlo (MC) simulation and series expansion have been used to determine some critical exponents. Based on the data available, a scaling form of the cluster distribution function has been verified. The fractal dimension D of the largest ('infinite' or 'spanning') cluster at p_c has also been determined and the cluster has been found to be nearly compact. An example of rotational constraint is obtained from the motion of an electron in the presence of a magnetic field perpendicular to the plane of motion. In spiral percolation, since motion is possible in both the forward direction and in a specific rotational direction, the appropriate example is the cycloidal motion of an electron in a disordered structure in the presence of crossed electric and magnetic fields. A quantitative study of the associated Hall effect may, however, require a more general version of the spiral percolation model. One important quantity in the percolation problem is the cluster external perimeter or 'hull' of the large clusters. The hull of a cluster is the continuous path of occupied sites at the external boundary of the cluster. The hull has a significance of its own apart from being a part of the percolation cluster. Sapoval *et al* (1985) have shown that the diffusion front resulting in diffusion from a source has a fractal structure that is related to the hull of percolation clusters. The percolation cluster hull in the case of undirected percolation exhibits scaling behaviour characterized by critical exponents which have values different from those of analogous quantities in the case of percolation clusters (Voss 1984, Weinrib and Trugman 1985, Ziff 1986). In this paper, we study the scaling behaviour of the hull in the case of spiral site percolation on the square and triangular lattices. In section 2, the scaling relations and the method used to obtain the scaling exponents are described. Section 3 contains the results obtained and a discussion of these results.

2. Scaling relations

This section contains the formulae and procedural details for calculation of some critical exponents to be defined below. The calculation of critical exponents has been done using the single-cluster growth method (Leath 1976, Alexandrowitz 1980) in which clusters are grown singly starting from a fixed origin. The size of the hull is given by the number of sites belonging to the hull and is determined by the method of Ziff *et al* (1984). The hull size distribution is given by

$$P_H(p) = N_H / N_{\text{tot}} \tag{1}$$

where N_H is the number of hulls of size H in a total number N_{tot} of clusters generated. Since each cluster has a hull associated with it, N_{tot} is also the total number of hulls generated. The various moments of $P_H(p)$, $\Sigma'_H H^k P_H(p)$ become singular as $p \to p_c$. The prime over the summation symbol denotes that the largest hull is excluded from the sum. In this paper we calculate the first two moments χ_H and χ'_H corresponding to $k=1$ and 2, respectively; χ_H gives the average hull size. As $p \to p_c$, χ_H and χ'_H are given by

$$\chi_H \sim |p - p_c|^{-\gamma_H} \qquad \text{and} \qquad \chi'_H \sim |p - p_c|^{-\gamma'_H} \tag{2}$$

where γ_H and γ'_H are the critical exponents. Formulae (1) and (2) are analogous to those defined for the percolation cluster (Santra and Bose 1992).

The fractal dimension of the hull of a percolation cluster has been determined using the method described by Ziff (1986). The position of every 64th occupied site is recorded, and this is done up to 1024 sites. Pairs from the list of points recorded are used to find the average distances for points separated by 64, 128, 192, ... sites. The distances denoted by R are averaged over the total number of clusters. The average of R is given by

$$\langle R \rangle \sim H^{1/D_H} \tag{3}$$

from which D_H, the fractal dimension of the hull, can be determined. D_H is related to the fractal dimension D of the percolation cluster by the relation

$$D_H / D = x \tag{4}$$

where x is obtained from the scaling relation

$$H \sim S^x. \tag{5}$$

S and H are the number of sites in the percolation cluster and the percolation cluster hull, respectively. The hull size distribution $P_H(p)$ is assumed to have a scaling form, as $p \to p_c$, similar to the cluster size distribution and is given by

$$P_H(p) = H^{-\tau_H + 1} f[(p - p_c) H^{\sigma_H}]. \tag{6}$$

The exponents τ_H and σ_H can be determined from the measured values of the exponents γ_H and γ'_H through the relations

$$\sigma_H = \frac{1}{\gamma'_H - \gamma_H} \qquad \text{and} \qquad \tau_H = \frac{3\gamma'_H - 4\gamma_H}{\gamma'_H - \gamma_H}. \tag{7}$$

The scaling behaviour described in equations (2)–(7) is similar to that for undirected percolation (Ziff 1986).

3. Results and discussion

The square and triangular lattices used for the simulation are of size 140×140 and 130×130, respectively. The values of the cluster exponents for spiral site percolation on the square lattice are already known (Santra and Bose 1992). The value of the correlation length exponent ν has, however, been wrongly quoted in the earlier work. In table 1, the correct value of ν as well as the values of the other exponents are given.

Table 1. Numerical estimates of cluster exponents for spiral site percolation on the square and triangular lattices.

Lattice type	Finite-size scaling				MC simulation		
	γ/ν	γ'/ν	D	β/ν	γ	γ'	ν
Square	2.01 ± 0.06	4.05 ± 0.13	1.957 ± 0.009	0.043 ± 0.009	2.19 ± 0.07	4.51 ± 0.16	1.116 ± 0.003
Triangular	1.97 ± 0.01	4.00 ± 0.02	1.969 ± 0.014	0.031 ± 0.014	2.23 ± 0.02	4.61 ± 0.05	1.136 ± 0.006

In the earlier paper, spiral site percolation on the triangular lattice has been considered with the constraint that only those turnings in the clockwise direction are allowed which have the least deviation from the original direction of motion. We now study spiral site percolation on the triangular lattice with turnings allowed in all clockwise directions. The values of $p_c(L)$ for lattices of size $L\times L$, where L ranges from 30 to 130, are given by $p_c(30)=0.585$, $p_c(40)=0.588$, $p_c(50)=0.589$, $p_c(60)=0.590$, $p_c(70)=0.591$, $p_c(80)=0.592$, $p_c(90)=0.592$, $p_c(100)=0.592$, $p_c(110)=0.593$, $p_c(120)=0.593$ and $p_c(130)=0.593$. The critical exponents are determined using the techniques of finite-size scaling and MC simulation, details of which can be obtained from the paper by Santra and Bose (1992). The values of p_c used for obtaining finite-size scaling and MC simulation data for various lattice sizes are the p_c values for respective lattice sizes. The values of the critical exponents determined are shown in table 1. The cluster and hull sizes have been recorded in bins; the ith bin contains sizes in the range 2^{i-1} to

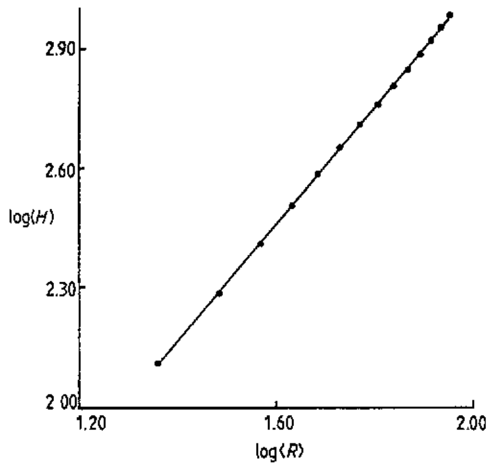

Figure 1. A plot of log(H) versus log(R) for the square lattice. From the slope, the value of the fractal dimension $D_H=1.476\pm0.005$ (equation (3)).

$(2^i - 1)$. The total number of clusters generated is 10 000. We now describe the results for the hull exponents for spiral site percolation on the square lattice. Figure 1 shows a plot of $\log(H)$ versus $\log\langle R\rangle$, which in accordance with (3) is found to be a straight line. From the slope of the straight line, the fractal dimension D_H of the hull is calculated as $D_H = 1.476 \pm 0.005$. The exponent x is obtained from relation (5) by plotting $\log(H)$ versus $\log(S)$. Figure 2 shows this plot, a straight line, from the slope of which the exponent x is determined as $x = 0.74 \pm 0.02$. From relation (4) and table 1, x is calculated as $x = 0.75 \pm 0.01$, which agrees within the limits of error with the measured value of x. Figures 3 and 4 give the plots of $\log(\chi_H)$ and $\log(\chi'_H)$ versus $\log|p - p_c|$. From the slopes of the straight lines (relation (2)) the exponents γ_H and γ'_H are determined as $\gamma_H = 1.82 \pm 0.01$ and $\gamma'_H = 3.75 \pm 0.02$. The errors quoted in the values of the exponents are the standard least-squares fit errors with the statistical

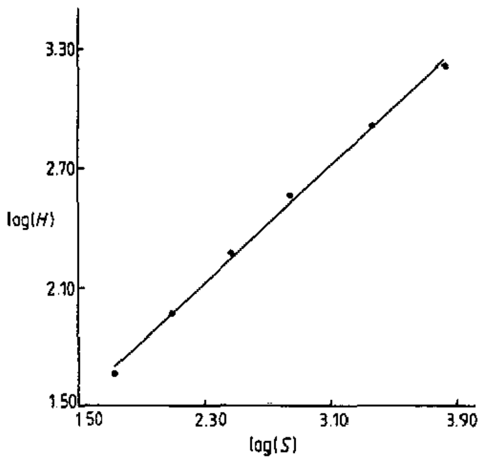

Figure 2. A plot of $\log(H)$ versus $\log(S)$ for the square lattice. From the slope, the value of the exponent x is $x = 0.74 \pm 0.02$ (equation (5)).

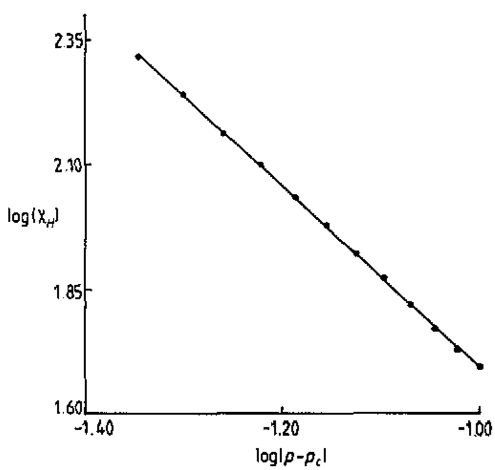

Figure 3. A plot of $\log(\chi_H)$ versus $\log|p - p_c|$ for the square lattice. From the slope, the value of the exponent γ_H is $\gamma_H = 1.82 \pm 0.01$ (equation (2)).

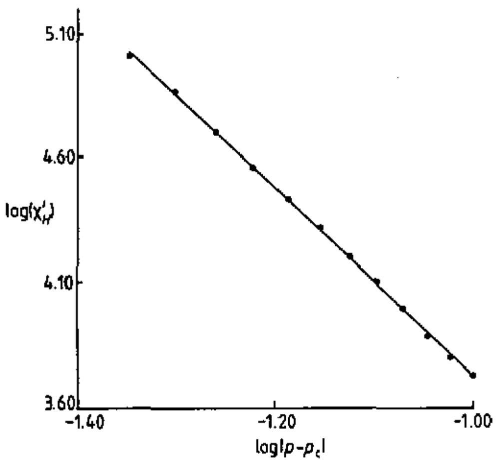

Figure 4. A plot of $\log(\chi'_H)$ versus $\log|p-p_c|$ for the square lattice. From the slope, the value of the exponent γ'_H is $\gamma'_H = 3.75 \pm 0.02$ (equation (2)).

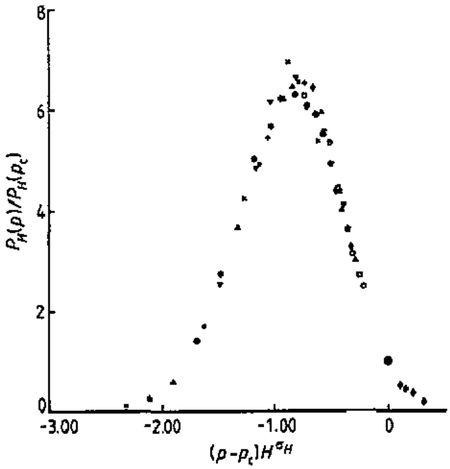

Figure 5. A plot of $P_H(p)/P_H(p_c)$ against $(p-p_c)H^{\sigma_H}$ for 13 different values of p with $\sigma_H = 0.518$ for the square lattice. $(p-p_c)$ has values in the range 0.015 to −0.110 with $32 < H < 512$. The symbol ⊕ indicates the position of the point (0, 1). The data plotted correspond to $p-p_c = 0.015$ (◆), −0.030 (○), −0.035 (□), −0.040 (△), −0.045 (◇), −0.050 (+), −0.055 (▽), −0.060 (×), −0.070 (●), −0.080 (▼), −0.09 (▲), −0.100 (☆) and −0.110 (■).

error of each data point taken into account. From equations (7), the exponents σ_H and τ_H are calculated as $\sigma_H = 0.518 \pm 0.008$, $\tau = 2.06 \pm 0.02$. A verification of the scaling function form in equation (6) is possible by plotting $P_H(p)/P_H(p_c)$ against $(p-p_c)H^{\sigma_H}$. If the scaling form is true, then, for sufficiently large clusters and for different values of p, the data should collapse on to a single curve. Figure 5 shows such a plot for 13 different values of p. The data for different values of p have been marked by different symbols, $(p-p_c)$ being in the range 0.015 to −0.110. The hull size H is within the limits 32–512. The symbol ⊕ denotes the location of the point (0, 1). The data collapse on to a single curve is found to be quite good, thus verifying the scaling function

Table 2. Numerical estimates of hull exponents for spiral site percolation on the square and triangular lattices.

Lattice type	D_H	x	γ_H	γ'_H	σ_H	τ_H
Square	1.476 ± 0.005	0.74 ± 0.02	1.82 ± 0.01 $D_H/D = 0.75 \pm 0.01$	3.75 ± 0.02	0.518 ± 0.008	2.06 ± 0.02
Triangular	1.466 ± 0.016	0.76 ± 0.02	1.91 ± 0.01 $D_H/D = 0.75 \pm 0.01$	3.87 ± 0.03	0.510 ± 0.010	2.03 ± 0.05

form (6). Table 2 displays the results, mentioned above, for the various exponents. An identical study is carried out for spiral site percolation on the triangular lattice. Figures 6–10 are in one-to-one correspondence with the figures 1–5 for percolation on the square lattice. The values of the exponents are given in table 2.

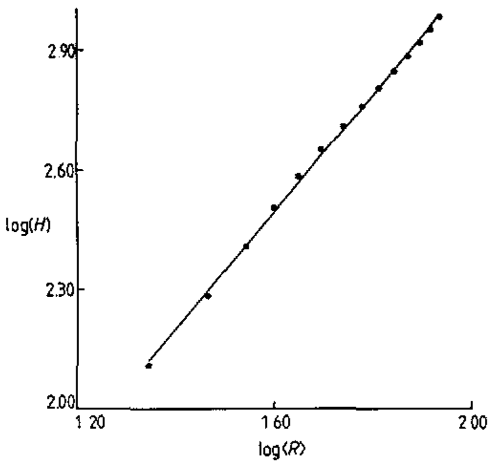

Figure 6. A plot of $\log(H)$ versus $\log\langle R\rangle$ for the triangular lattice. From the slope, the value of the fractal dimension $D_H = 1.466 \pm 0.016$ (equation (3)).

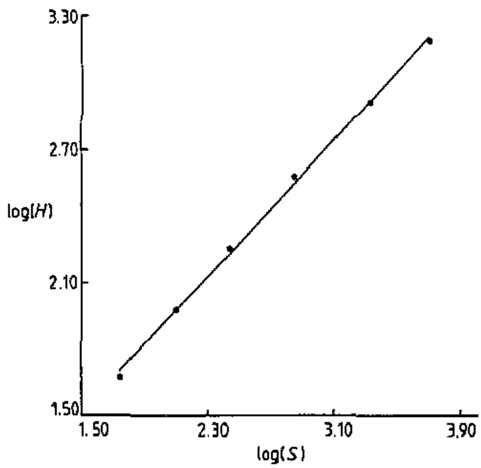

Figure 7. A plot of $\log(H)$ versus $\log(S)$ for the triangular lattice. From the slope, the value of the exponent x is $x = 0.76 \pm 0.02$ (equation (5)).

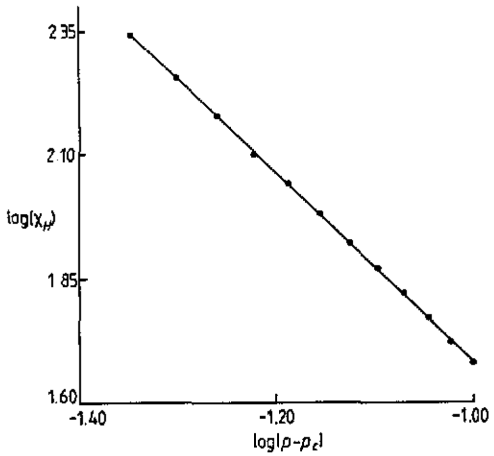

Figure 8. A plot of $\log(\chi_H)$ versus $\log|p-p_c|$ for the triangular lattice. From the slope, the value of the exponent γ_H is $\gamma_H = 1.91 \pm 0.01$ (equation (2)).

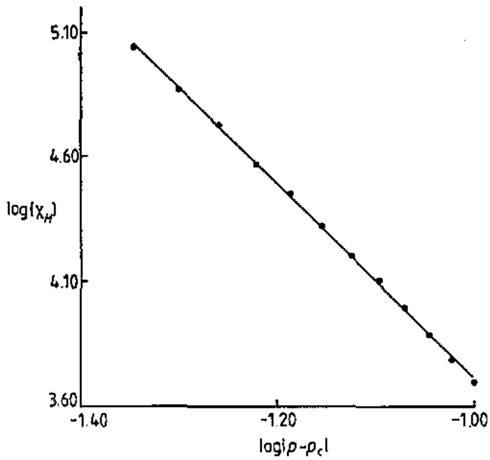

Figure 9. A plot of $\log(\chi'_H)$ versus $\log|p-p_c|$ for the triangular lattice. From the slope, the value of the exponent γ'_H is $\gamma'_H = 3.87 \pm 0.03$ (equation (2)).

By examining tables 1 and 2 we reach the following conclusions. From the values of the fractal dimensions calculated for both the cluster and the hull, the cluster appear to be nearly compact whereas the hulls are fractal. Figures 11(a) and 11(b) show a typical spiral percolation cluster and its hull on a square lattice of size 100×100. Clusters and hulls have a similar behaviour but the critical exponents have different magnitudes. The exponents σ_H and τ_H have similar values within the margin of error for both the square and triangular lattices. The last two results are in agreement with the scaling behaviour characteristic of undirected percolation. In this case, the clusters are also fractal objects and are not nearly compact as in the case of spiral percolation. The values of γ_H and γ'_H quoted in table 2 do not seem to agree within the limits of error for the square and triangular lattices. Apparently, the errors have been underestimated. The expressions (7) for σ_H and τ_H involve differences in the values of γ'_H and

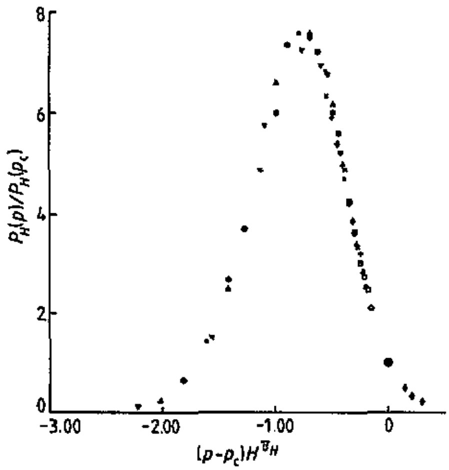

Figure 10. A plot of $P_H(p)/P_H(p_c)$ against $(p-p_c)H^{\sigma_H}$ for 13 different values of p with $\sigma_H = 0.510$ for the triangular lattice. $(p-p_c)$ has values in the range 0.015 to -0.110 with $16 < H < 512$. The symbol $\oplus$ indicates the position of the point $(0, 1)$. The data plotted correspond to $p - p_c = 0.015$ (◆), -0.30 (○), -0.035 (□), -0.040 (△), -0.045 (◇), -0.050 (+), -0.055 (×), -0.060 (▽), -0.070 (☆), -0.080 (■), -0.090 (●), -0.100 (▲) and -0.110 (▼).

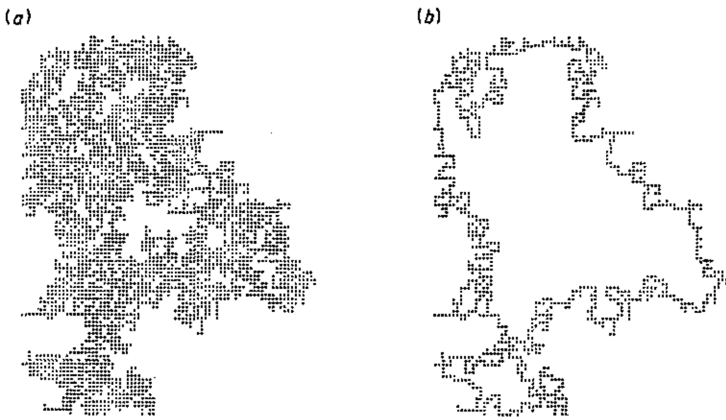

Figure 11. Example of (a) a spiral percolation spanning cluster and (b) its hull on a square lattice of size 100×100 at $p_c = 0.7083$.

γ_H, which lead to an agreement of values of σ_H and τ_H within the limits of error for both the square and triangular lattices. For undirected percolation, Sapoval *et al* (1985) have proposed the relation $D_H = 1 + 1/\nu$ between the fractal dimension D_H of the hull and the correlation length exponent ν. This prediction is supported by direct measurement (Voss 1984, Kremer and Lyklema 1985). From tables 1 and 2 one can verify that the conjectured relation does not hold true for spiral site percolation and so seems to be characteristic only of undirected percolation.

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